8 The Logic in Dedekind's Logicism

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Any history of logic from Kant to the 20th century needs to take into account the emergence of logicism, since it is largely in this context that logic as we know it became prominent. Usually Gottlob Frege and Bertrand Russell are seen as its two main representatives, especially early on, and the more recent rise of neo-logicism, in writings by Crispin Wright, Bob Hale and others, proceeds along Fregean lines as well. In this chapter, I will focus on Richard Dedekind instead. Today Dedekind is sometimes mentioned in connection with logicism; yet at the end of the 19th century, he was the most prominent logicist, ahead of Frege. This alone invites further reflection. Reconsidering Dedekind also leads naturally to questions about what was, or could be, understood by "logic" in this context. Addressing them will involve developments in 19th-century mathematics that are relevant in two respects: by forming core parts of the background for the emergence of modern logic; and by pointing towards a distinction between two conceptions of logic that deserve more attention, both historically and philosophically.[1]

The chapter proceeds as follows. First, I will document Dedekind's characterization of his project as showing that arithmetic, understood in an inclusive sense, is "part of logic", and I will put that project into the context of broader developments in 19th-century mathematics. In the second section, I will look in more detail at Dedekind's procedure, including the fact that it involves set-theoretic "constructions" and a certain kind of "abstraction". Third, a brief summary of later appropriations and developments of Dedekind's contributions will be provided, especially in axiomatic set theory and category theory, although these are usually not seen as forms of "logicism". This will,

1 An early version of this chapter was presented at McMaster University, Hamilton, Canada, in May 2016. A later version formed the basis for a talk at the CSHPM/SCHPM meeting in Toronto, Canada, May 2017. I would like to thank the audiences at both events for their comments. I would also like to thank Sandra Lapointe for inviting me to them, as well as for comments on the later written version. This chapter builds on Reck (2013a, 2013b); there is also partial overlap with Reck (forthcoming), Reck and Keller (forthcoming), and Ferreirós and Reck (forthcoming).

in the fourth section, provide the background for the question of how Dedekind—and parallel to him, Frege—must have understood "logic" for their logicist projects to make sense, namely in a wider sense than the one dominant in the 20th century. The essay will conclude with some general observations, both about the contested status of the notion of "logic", i.e. a lack of consensus about its nature that is still often underestimated, and about its historical and philosophical relationship to modern mathematics.

Dedekind's Logicism: Programmatic Remarks and Historical Background

The text in which Dedekind's logicism is most explicitly stated is his well-known booklet *Was sind und was sollen die Zahlen?* (Dedekind 1888), which builds on his earlier *Stetigkeit und irrationale Zahlen* (1872). In both texts, he addresses issues concerning the foundations of "arithmetic" understood in a broad sense, from the theory of the natural numbers to traditional algebra and higher analysis, including the real numbers (and even the complex numbers in the end, although this is less explicit). In the Preface to his 1888 text, Dedekind adds programmatically that he will develop "that part of logic which deals with the theory of numbers" (Dedekind 1963, p. 31); this is elaborated further as follows:

> In speaking of arithmetic (algebra, analysis) as part of logic I mean to imply that I consider the number concept entirely independent of the notions of intuition of space and time, that I consider it an immediate result of the laws of thought
>
> (ibid.).

Dedekind never uses the term "logicism" himself (nor does Frege). As such remarks show, his goal is nonetheless to establish that arithmetic is "part of logic", and that involves relying solely on "the laws of thought" while rejecting any dependence on our "intuition of space and time".

There are clear echoes of Kant's philosophy in these remarks. This is so already by how the choice is framed, namely as one between using only logical laws or appealing to intuition as well, and even more, by characterizing the latter in terms of Kantian spatio-temporal intuition. One of the main reasons for Dedekind's refusal to appeal to space and time in this context is more mathematical, however. As he writes,

> It is only through the purely logical process of building up the science of number and by thus acquiring the continuous number domain that we are prepared to accurately investigate our notions of space and time [...]
>
> (*ibid.*, pp. 31–2).

In other words, we need a precise, prior account of the real numbers to understand space and time accurately, not *vice versa*. But what does Dedekind mean by "the purely logical process of building up the science of number"? Also, what are the "laws of thought" on which his approach is to be based, including the philosophical framework in the background, Kantian or otherwise? I will address the former questions first, including putting Dedekind's corresponding contributions in historical context. We will return to the latter question in later sections.

The crucial and arguably most innovative step in Dedekind's "purely logical process of building up the science of number" had already been presented in his 1872 booklet. It consists in the construction, by means of his notion of cut (Dedekind cut), of the real numbers out of the rational numbers. This step was crucial since it provided the missing link in a series of "domain extensions", from the natural numbers through the integers and rationals to the complex numbers. It was also the most innovative step, since all the others can be done in terms of pairs of numbers (or more precisely, equivalence classes of such pairs), e.g. rational numbers can be understood basically as pairs of integers etc. The main model in this connection was W.R. Hamilton's account of the complex numbers in terms of pairs of reals, which built on Gauss' earlier introduction of the complex number plane. Clearly aware of Hamilton's and Gauss' works (Gauss was his dissertation advisor), Dedekind also knew of parallel introduction of the rationals and integers based on the natural numbers.[2] The step from the rationals to the reals is different, because it involves the infinite in a more substantive way (see later).

This process of building up the familiar number domains, or the step-by-step reduction from the complex numbers all the way down to the natural numbers, is often seen as part of the "arithmetization of analysis". Dedekind takes on this perspective as well, e.g. when he talks about showing that "every theorem of algebra and higher analysis, no matter how remote, can be expressed as a theorem about the natural numbers", a project he associated with his mentor Dirichlet (Dedekind 1963, p. 35). In addition, the process is usually seen as driven by the elimination of infinitesimals in the Calculus, i.e. by rethinking the notion of limit in terms of the ε-δ-method, which ultimately relies on a precise, unified account of the rational and real numbers, precisely as provided by Dedekind. However, one can understand the arithmetization project also as part of demonstrating—and this is what Dedekind did, as we saw—that higher analysis is independent of the notions of space and time. Central to this demonstration is replacing the notion of magnitude, or

2 Cf. Sieg and Schlimm (2005), Reck (2016), and Ferreirós and Reck (forthcoming) for more details. Dedekind discussed this series of domain extensions already in Dedekind (1854).

of "measurable quantity", by the concept of real number (which allows for a related account of complex numbers too). And closely related with that replacement is the shift from relying on "intuitive" geometric evidence about magnitudes towards more precise and explicit "conceptual" reasoning (as illustrated, e.g. by Dedekind's proof of the Mean Value Theorem in *Stetigkeit und irrationale Zahlen*).

From a broader perspective, the following three developments in 19th-century mathematics should be seen as crucial elements of the background and motivation for Dedekind's foundational contributions. First, there is the shift away from taking geometry to be the ultimate basis for all of mathematics (a position one might call "geometricism", grounded in traditional geometric constructions and intuitive evidence) towards seeing arithmetic, in the aforementioned inclusive sense, as being independent and more basic (thus "arithmetizing" algebra and analysis).[3] Second, there is the push towards founding reasoning in mathematics, and in arithmetic especially, on explicitly defined concepts and logical derivations from them, as opposed to relying either on intuitive geometric considerations or on "blind calculation" (the adoption of a more "conceptual" methodology, also opposed to certain forms of formalism).[4] Third, there is the move towards reconstructing crucial, and often novel, mathematical entities—not only the real numbers, but also "ideals" in algebraic number theory (another of Dedekind's main contributions), "transfinite numbers" in Georg Cantor's work, and "points at infinity" in projective geometry—by using (what we would call) set-theoretic constructions, often involving infinite sets essentially.[5] All three of these developments were picked up and continued in Dedekind's works, in several cases by providing capstone contributions to the relevant fields. Finally and crucially for our purposes, with the set-theoretic treatment of the natural numbers in his 1888 essay, he pushed them a significant step further, thus in effect basing all "pure mathematics" of his time on "laws of thought" alone.[6]

Seen from this perspective, Dedekind's "logicism" thrives on a unification and systematic extension of broader developments in 19th-century mathematics. Since many mathematicians at the end of the century valued these developments highly, his relevant contributions and

3 In this connection, one can speak of the "birth of pure mathematics, as arithmetic" in the 19th century; cf. the title of, and the further discussion in, Ferreirós (2007).

4 Howard Stein and others have talked about the 19th-century birth of a kind of "conceptual mathematics" in this connection; cf. Stein (1988), also Reck (2013a, 2016).

5 With respect to this third point, seen as originating in the case of geometry, Mark Wilson has talked about the rise of "relative logicism" in the 19th century; cf. Wilson (2010).

6 For more illustrations and a further defense of this perspective, cf. Reck (2013a, 2016).

programmatic remarks did not go unnoticed. Thus, Ernst Schröder, the foremost German member of the Boolean school of algebraic logic, wrote of being tempted to join "those who, like Dedekind, consider arithmetic a branch of logic". Similar remarks can be found in David Hilbert's early works.[7] Within philosophy, the neo-Kantian Ernst Cassirer adopted Dedekind's logicism in the early 20th century.[8] And C.S. Peirce, who did not see himself as part of the logicist camp, acknowledged Dedekind as someone who "holds mathematics to be a branch of logic" (Peirce 1902, p. 32). By contrast, Frege's works were much less widely known at the time, partly because his contributions to mainstream mathematics were more minor. It took Russell's later appropriation of Fregean logicism to bring it to broader attention.

Two Sides of Dedekind's Logicism: Construction and Abstraction

A core ingredient of Frege's and Russell's versions of logicism is their introduction of a logical language and a corresponding formal calculus, i.e. a deductive system for higher-order logic. Each of them presented these aspects, in their respective versions, explicitly and in detail. They also advertised them as crucial advances over the logic of their predecessors, especially traditional Aristotelian logic. Nothing comparable can be found in Dedekind's works. This is one reason why he is sometimes not put on the same level, with respect to either logicism or the rise of mathematical logic. But a second, equally important ingredient of logicism, present also in Frege's and Russell's versions, consists in the introduction of a theory of sets, extensions, or classes. In fact, the use of such a theory lies at the core of the logicist reconstruction of the natural and the real numbers. In light of that fact, it is worth comparing Dedekind's approach to theirs in some detail. In addition, for Dedekind, logicism involves not only certain set-theoretic constructions, but a form of "abstraction" that distinguishes his version sharply from Frege's and Russell's. To get a better sense of the resulting, distinctive logicist position, including what is meant by "logic" in it, both processes have to be taken into account.

What are the main set-theoretic constructions used in Dedekind's logicist project? I mentioned earlier the one central to *Stetigkeit und irrationale Zahlen*, namely the introduction of cuts in the system of rational numbers. A closer look reveals that there are three main steps involved in it. First, Dedekind starts by considering all the rational numbers together,

7 For Hilbert, see Ferreirós (2009); for Schröder and more generally, compare Reck (2013a, 2013b).

8 Cf. Cassirer (1910), especially Ch. II; for further discussion, see Reck and Keller (forthcoming).

seen as an infinite system (an ordered field that contains all the natural numbers). In a second step, he considers cuts on that system in the usual mathematical sense, where each cut consists of two infinite sets. Third, he introduces the system of all such cuts, endows it with an ordering relation and arithmetic operations (induced by those on the rational numbers), and shows that the result is continuous (a line-complete ordered field). What makes the third step especially noteworthy is that it implicitly involves the full power set of the set of the rational numbers (an application of the Power Set Axiom), which leads from a countable to an uncountable set (as Cantor would soon establish). In that respect, the reconstructions of the integers and rational numbers in terms of pairs are less substantive. On the other hand, the latter involve the notion of pair as a basic "logical" ingredient. Such details are worth noting in our context, since they reveal what the notion of "logic" at play involves.[9]

Turning to the treatment of the natural numbers in *Was sind und was sollen die Zahlen?*, which set-theoretic constructions and basic notions play a role in it? Before answering that question, let me make a more general observation that highlights Dedekind's originality. A striking aspect of his 1888 essay is that it starts with a general framework of sets ("Systeme") and functions ("Abbildungen"), both understood as allowing for arbitrary cases (not only sets and functions involving elements, arguments, and values of all kinds, but also non-decidable sets and functions). While Dedekind builds on the generalized notion of function introduced by his mentor Dirichlet a few years earlier, together with the novel use of sets in Cantor's work, this was a radical innovation. In fact, Dedekind seems to have been the first person to propose using such a framework for systematically rethinking the foundations of arithmetic in the inclusive sense, and thus, the foundations for all "pure mathematics" at the time. He was also one of the first to treat sets and functions extensionally (assuming an Axiom of Extensionality for both). Finally, he considered his general notions of set and function, together with the framework to which they belonged, a part of "logic".[10]

Within such a "logical" framework, there are then several important steps in Dedekind's reconstruction of the natural numbers in his 1888 essay. First, he defines what it means for a set to be infinite (Dedekind-infinite, i.e. 1-1 mappable onto a proper subset). Then he defines the notion of a "simple infinity" (basically, the minimal closure of

9 Today we are used—from axiomatic set theory—to reducing the notion of pair to that of set (following Wiener or Kuratowski); but Dedekind does not suggest such a reduction. Nor does he reduce functions to sets (of tuples). More on both aspects later.

10 While not explicit in Dedekind (1888), this is clear from an 1887 draft of it. There he notes that the theory of sets, or of "systems of elements", is "logic"; cf. Ferreirós (1999), p. 225. Here and at various other points, I am strongly indebted to Ferreirós work, including Ferreirós (forthcoming).

a singleton set under a 1-1 function, thereby using a minimality clause equivalent to Peano's induction axiom). Next, the latter notion is shown to provide the basis for a "logical" treatment of mathematical induction (just as Frege had done by using his notion of the "ancestral relation"), and even for an explicit, systematic justification of recursive definitions and inductive proofs much more generally. After that come two core theorems in Dedekind's procedure, concerning (i) the existence of an infinite set, thus also of a simply infinite set, and (ii) the fact that any two simple infinities are isomorphic (so that the notion of simple infinity is categorical). As Dedekind adds, together these justify taking any simply infinite set to "play the role" of the natural numbers, in the sense that "translations" of all theorems concerning the natural numbers will hold for each of them (Dedekind 1963, pp. 95–6).

With respect to his overall procedure, but especially the last few steps just noted, several features call for further comment. The first concerns Dedekind's proof, or attempted proof, of the existence of an infinite set (his Theorem 66). What he appeals to in this connection are the following ingredients: "the totality S of all things which can be objects of my thought" (a kind of universal set); a 1-1 function f on S which maps any element s onto "the thought s', that s can be object of my thought" (serving as a successor function on S); and Dedekind's "ego" or "self" (a base element of S different from all values of the function f). The suggestion is, basically, to start with a distinctive element a of S, such as Dedekind's "self", and to construct a simple infinity by closing $\{a\}$ under f in S.[11] As should be added, this procedure for the case of the natural numbers is parallel to Dedekind's 1872 construction of the system of all cuts in the rational numbers for the introduction of the reals. In the latter case, what we get is the construction of a complete ordered field; here we get the construction of a simply infinite set.

Dedekind's procedure in his 1888 essay, and especially the aspects just highlighted, were seen as problematic from early on. The main reason is that his use of a universal set leads directly to paradoxes (such as Russell's, Burali-Forti's, etc.), as Cantor informed him in the late 1890s. Beyond that, his construction of a simple infinity appears to depend on blatantly "non-mathematical" entities, namely "thoughts" and Dedekind's "self", as various critics complained, starting with Russell in 1903.[12] Dedekind acknowledged the former as a serious problem as soon as he found out about it. But is the latter really as damning as often assumed? Note that it is not hard to substitute Dedekind's original base element by the empty set, $\emptyset$, and his original successor function by the function that maps x

11 See Dedekind (1963), p. 64. Dedekind's particular choices of base object and successor function were obviously meant to assure that the outcome would be a simple infinity.

12 Cf. Reck (2013b) for further discussion of Dedekind's reception.

onto $\{x\}$, as suggested by Zermelo. Or we can start with Ø and use von Neumann's successor function, $x \rightarrow x \cup \{x\}$, as is standard in axiomatic set theory today. Dedekind himself did not suggest such substitutions. But both would seem to be consistent with his approach, even within what he considered "logic". Then again, even if we allow for them, there is still a question. Namely, what guarantees the existence, not only of the set Ø and each of its successors, but also of the set containing all of them? Dedekind's simple infinity is constructed as a subset of the universal set S (implicitly using a Separation Axiom), and the latter is problematic. Moreover, parallel questions arise concerning the existence of various functions and relations used by Dedekind along the way.

Another feature of Dedekind's 1888 procedure that calls for further clarification involves not set-theoretic "construction" but structuralist "abstraction". As noted earlier already, on his account, any simple infinity can "play the role" of the set of natural numbers. However, Dedekind does not leave it at that; he suggests the following further step (Remark 73). Starting with any simple infinity, e.g. one of those mentioned earlier (it doesn't matter which one, since they are all isomorphic), we "entirely neglect the special character of the elements, simply retaining their distinguishability and taking into account only the relations to one another"; that is, we perform an abstraction that "frees the elements of every other content". The result will be "the natural numbers", now understood as a separate, distinguished simple infinity (whose elements are characterized "purely structurally", as one might add). Finally, Dedekind calls the resulting numbers "a free creation of the human mind" (Dedekind 1963, p. 68).

From the standpoint of 20th-century axiomatic set theories (e.g. ZFC, the Zermelo-Fraenkel axioms with the Axiom of Choice), it is tempting to downplay or ignore Dedekind's appeal to "abstraction" and "free creation". It is also true that they are not fully clarified in his 1888 essay. At the same time, a parallel appeal to "creation" (although not yet to "abstraction") occurs in Dedekind's 1872 essay. There too, he does not simple want to have the cuts on the rational numbers to "play the role" of real numbers, as is standard procedure today. Rather, he insists on introducing novel, separate, and "pure" objects determined by them; and again, it is the latter that deserve to be called "the real numbers". Finally, in correspondence from the 1880s, Dedekind insists that his introductions of the real numbers and the natural numbers are meant to involve "abstraction" and "creation" in the same sense.[13]

The present essay is not the place to fully explore what is, or could be, going on in Dedekind's appeal to "abstraction". Nonetheless, this question needs to be addressed at least to some degree, since it concerns

13 Insisting on this point, i.e. on this reading of "Dedekind abstraction" for both his 1872 and 1888 essays, is not uncontroversial. Cf. Reck (2003) for a further defense. In Sieg and Morris (forthcoming), an alternative is presented (one in which Dedekind's 1888 essay is read as closer to set-theoretic practice).

one side of his logicism. In particular, the question needs to be raised in which sense, if any, not just the set-theoretic constructions used by him but also the kind of "Dedekind abstraction" just described could possibly be seen as part of "logic". The latter is even more urgent because Dedekind abstraction was quite unpopular for much of the 20th century. Indeed, it was either dismissed as incoherent (from Russell to Michael Dummett and beyond) or simply ignored (in set-theoretic reconstruction of Dedekind's foundational contributions).[14] Consequently, a discussion of its "logicality" is long overdue.

From Dedekind's Logicism to Axiomatic Set Theory and Category Theory

The most positive and detailed reception of Dedekind's foundational contributions occurred in axiomatic set theory, starting with Ernst Zermelo's work in the early 20th century (Zermelo 1908, etc.). Rather than dismissing Dedekind's approach as based on an inconsistent theory, what Zermelo did was to carefully reconstruct which set-theoretic constructions are involved so as then to reformulate the background requirements as "axioms". I already mentioned three of them parenthetically: the Axiom of Extensionality, the Power Set Axiom, and the Axiom of Separation. Zermelo also turned Dedekind's often vilified argument for the existence of an infinite set into the now standard Axiom of Infinity (even calling it "Dedekind's Axiom"), by using the sequence Ø, {Ø}, {{Ø}}, ... instead of Dedekind's original simple infinity. A few further basic steps involved in Dedekind's works, corresponding to Boolean operations on sets (unions, intersections, and set-theoretic differences), can be covered by the Axiom of Separation as well, while for others the Axiom of Unions and the Axiom of Pairing were introduced.

In subsequent works, it was realized that certain steps taken by Dedekind implicitly involve the Axiom of Choice (aspects of his treatment of the notions of "finite" and "infinite") and the Axiom of Replacement (his general treatment of recursion and induction etc.), so that these had to be added as well. On the other hand, the notion of pair does not need to be assumed as basic, as Dedekind had done, since we can reconstruct it set-theoretically (along familiar Kuratowskian lines). And that reconstruction does not just affect the construction of the integers and rational numbers (as well as the complex numbers); it allows for a general reduction of the notion of function to that of set (by considering sets of tuples). Hence, we can work with one basic notion alone, namely that of set. Finally, an Axiom of Foundations can be added, since non-well-founded sets do not play an essential role in classical mathematics. And with this list of axioms in place, Dedekind's "logicist" constructions can

14 For a further discussion of such negative reactions, cf. Reck (2013b).

simply be repeated in axiomatic set theory, while avoiding antinomies such as Russell's, Burali-Forti's, etc.

But does this rehabilitate Dedekind's logicism? It clearly saves (almost all of) his technical results, and fairly directly so. Hence it allows for a "reduction" of all the pure mathematics with which he dealt to axiomatic set theory. Note also the following remark by Zermelo, which most likely reflects Dedekind's influence as well:

> Set theory is that branch of mathematics whose task is to investigate mathematically the fundamental notions of 'number', 'order', and 'function', taking them in their pristine form, and to develop thereby the logical foundations of all of arithmetic and analysis
>
> (Zermelo 1908, p. 200).

While Zermelo speaks of the "logical foundations" of mathematics in this passage, usually axiomatic set theory is not considered a form of logicism; similarly, Zermelo is not counted as a logicist. This is for several related reasons. First, the notion of set codified in its axioms tends not to be seen as purely "logical", e.g. because it involves the intuition behind the standard "cumulative" conception of set.[15] Second, set theory is usually taken to be another mathematical theory (as Zermelo does in the previous quote, too) as opposed to a separate "logical" basis, even though it is fundamental insofar as we can use it to interpret other mathematical theories within it. Third and perhaps most importantly, the existence claims crucial for set theory are no longer seen as substantive "logical truths" today, but as captured in terms of "axioms" understood in a sophisticated "formalist" way (along Hilbertian lines). And as a consequence, Zermelo's reconstruction of Dedekind's approach is taken not as rehabilitating logicism, but as reconceiving them in terms of formalism. Having said that, it is not clear that ZFC has to be understood formalistically.[16]

A second 20th-century approach that allows for a rehabilitation of Dedekind's technical results is provided by category theory. In certain respects, this is actually closer to Dedekind's original procedure, e.g. by taking the notion of pair as basic (in terms of the notion of Cartesian product) and by not reducing the notion of function to that of set (indeed, by treating functions as more basic than sets). Moreover, Dedekind's

15 In addition, the "unconstrained" and "non-conceptual" nature of the Power Set Axiom may be seen as rendering it "non-logical". Then again, it would be interesting to explore if adopting Gödel's "V=L" could be taken to restore the "logicality" of the theory. Gödel did, after all, arrive at his constructive universe by starting from Russell's logicist type theory.

16 Note also that axiomatic set theory is often grouped together with proof theory, model theory, etc., under the label "mathematical logic", thereby using some general sense of "logic".

approach to induction and recursion, thus his treatment of the natural numbers, finds an elegant and fruitful home in this context (via the notion of universal mapping properties, etc.).[17] Yet like in the case of axiomatic set theory, category-theoretic approaches tend not to be seen as forms of "logicism". Typically they are understood along Hilbertian formalist lines as well, e.g. by assuming different category-theoretic axioms as basic for different purposes and treating them formalistically.[18]

There is a certain view of "logic" that typically goes together with the adoption of formalism along such lines, a view that became widely accepted in the 20th century (but arguably has roots in Kant too). According to this view, what characterizes "logical truths" is that they are true "in all domains"—they are something like "tautologies" in Wittgenstein's sense, "analytic truths" in Carnap's sense, or "true in all models" in Tarski's sense. An immediate consequence of this view is that logic, by itself, cannot prove or otherwise justify any existence claims, including mathematical existence claims.[19] Instead, it is precisely "axioms" understood formalistically that are used for this purpose (axioms for whose consistency we can, within limits, argue in meta-logical ways, i.e. model-theoretically, proof-theoretically, etc.).

While this view is widespread, the following observation should be added. Along such lines, logicism turns out to be simply a nonstarter—not just the logicism of Dedekind, but also Frege's logicism and, say, the neo-logicism of Wright, Hale, etc.[20] Has logicism then been undermined decisively? Only if the notion of "logic" just described, together with some version of formalism, is inevitable. More basically, what this line of thought indicates is that Dedekind, Frege, and related thinkers worked with a different, less restrictive notion of "logic". And that notion remains in need of clarification, both historically and systematically.

Dedekind and Frege on Basic Logical Notions and Logical Laws

Let me restrict our discussion to the logicists Dedekind and Frege at this point, so as not to make things overly complicated. As has often been acknowledged, Frege was neither fully explicit nor entirely clear in his

17 Cf. McLarty (1993), or more generally the literature referred to in it.

18 Then again, various versions of category theory can be reconstructed in "mathematical logic" too, e.g., by framing them type-theoretically.

19 Or in Kant's earlier terms, "logic" (in the sense of "general logic", as opposed to "transcendental logic") does not involve any reference to specific objects, or indeed, to objects at all.

20 Insofar as Russell's mature logicism involves a "no-classes" theory of classes, and hence, an elimination of them as logical objects, his case is more complicated (also by involving the status of the Axiom of Reducibility and the Axiom of Infinity).

writings about what makes a notion or truth "logical". But roughly, it is the "generality" of logic that seems crucial for him. Logical notions are those used in all reasoning (or perhaps all "exact" reasoning); and logical truths are those that "govern" all objects, concepts, and functions. The contrast is with the notions and truths of the special sciences. For example, the truths of geometry "govern" geometric entities, such as points, lines, etc., and only those; similarly, the truths of mechanics "govern" physical entities and processes. The truths of logic, on the other hand, are not restricted that way. Again, logic provides a framework for all (exact) reasoning, in terms of its basic notions and its basic laws.

Dedekind is as sparse as Frege, or more so, in his discussion of what makes a notion or a truth "logical". But he seems to conceive of them in a way not too different from Frege, in several respects. First, for both the basic notions of logic include those of function and set/class (even if the two thinkers do not spell them out in exactly the same way). Second, for Dedekind, like for Frege, logic "governs" all reasoning, in particular all reasoning in mathematics. In fact, without logic, such reasoning would be impossible for him. Put differently, logical notions and truths are seen as indispensable by him. Interestingly, Dedekind's most explicit statement along such lines concerns the notion of function; as he writes:

> If we scrutinize closely what is done in counting a set or a number of things, we are led to consider the ability of the mind to relate things to things, to let a thing correspond to a thing, or to represent a thing by a thing, an ability *without which no thinking is possible.* Upon this unique and therefore absolutely indispensable foundation the whole science of number must, in my opinion, be established
> (Dedekind 1963, p. 32, my emphasis).[21]

With respect to sets, or "systems", Dedekind is less emphatic; he just notes:

> It *very frequently happens* that different things a, b, c, ... can be considered from a common point of view [...] and we say that they form a system S [...]
> (*ibid.*, my emphasis).

21 A similar remark occurs in Dedekind's well-known letter to Keferstein, where he characterizes the core of his project in the 1888 essays as follows:

> What are the mutually independent properties of the sequence of numbers N, that is, those properties that are not derivable from one another but from which all others follow? And how should we divest these properties of their specific arithmetic character so that they are subsumed under more general concepts and under activities of the understanding, without which no thinking is possible at all [...].
> (Dedekind 1890, pp. 99–100)

Then again, the notion of set/class is so closely intertwined with the notion of function for Dedekind (e.g. both the domain and the range of functions are sets for him) that the latter is indispensable by implication as well, as one might assume.

Frege would, of course, not want to talk about "the mind" in connection with logic, since this invites psychologistic confusions. Yet one does not have to understand Dedekind's appeal to the "mind" in a subjectivist, individualistic sense, the sense Frege would find especially objectionable. To bring the two even closer together, one might also replace Dedekind's appeal to "thinking" by a more objective conception of "thought" (as Frege himself suggested). Furthermore, both thinkers work within a Kantian epistemological framework, even if they disagree with Kant about how arithmetic fits into it.[22] In fact, in Dedekind's case, the appeal to "abilities of the mind" might be seen as pointing towards Kant's "categories of the understanding", in the sense that the notion of function should count as such a category. "Logic" is then the discipline that deals with presuppositions for all thinking; and the ability to "think functionally" is an important, so far underemphasized example.[23]

In any case, for Dedekind, and in a related way for Frege, "logic" includes a general framework of functions and sets/classes. Within such a framework, we can reconstruct the basic notions and principles of arithmetic in the broad sense, thus all the "pure mathematics" of the late 19th century. And we can do so without appealing to intuitive considerations in the traditional geometric sense, as Dedekind insisted in his two foundational essays and as was crucial for Frege too. Put in more traditional terminology, what "logic" provides in our context is the framework in which we can reconstruct the *logos* of pure mathematics—its basic notions and laws—in an explicit, systematic way.[24] So much seems clear about Dedekind. Less clear is what exactly the relevant logical principles are supposed to be. One would expect him to have made explicit his basic "laws of thought", not just his basic "logical notions". But that is not the case, i.e. he never formulated such laws explicitly. In that sense, both the constructions and the abstraction in Dedekind's writings remain without precise backing.

22 In my account, the parallels between Frege's and Dedekind's logicisms are emphasized, i.e. their two perspectives are assimilated, like on this point. For an interesting approach that highlights the differences much more, cf. Benise-Sinaceur et al. (2015). I plan to respond to the latter in a future publication.

23 Cf. Klev (2017) for a recent interpretation of Dedekind's conception of logic along Kantian lines. For a related reading that is more neo-Kantian, cf. Reck and Keller (forthcoming). Note that along both lines "logic" includes Kantian "transcendental logic", not just his "general logic".

24 With respect to this appeal to "logos", I have in mind Marburg Neo-Kantianism as exemplified by Cassirer and his two teachers, Hermann Cohen and Paul Natorp; cf. again Reck and Keller (forthcoming).

This lack of explicit basic laws was Frege's main complaint about Dedekind's logicist project, especially in Frege's *Grundgesetze der Arithmetik, Vol. I* (1893).[25] And as already acknowledged, Dedekind did not specify a logical language and a related deductive calculus either. Concerning the former, in hindsight it seems natural to reconstruct his approach either in set-theoretic language or, more directly (since functions are basic for him), in type-theoretic language, and particularly, in the language of a simple theory of types (because of his extensional conception of sets and functions). Now, earlier we considered Zermelo's reconstruction of the basic construction principles needed for Dedekind's purposes in terms of set-theoretic axioms. In contrast, all one can find in Dedekind's own writings is the use of a general comprehension principle for sets; or better, he seems to assume a universal set together with a general principle for forming subsets (a general separation principle).[26] But even that much has to be reconstructed from his general procedures, i.e. he does not make such laws explicit himself.

Why did Dedekind not formulate basic "logical laws" explicitly? It is hard to be sure. Perhaps he was simply the first to work with a general framework of sets and functions for foundational purposes, so there was no precedent for it. (Boole and his followers had formulated some laws for classes, but in a more restricted way and not for foundational purposes.[27] Frege's relevant work was slightly later.) Or Dedekind assumed an older conception of "logic", one according to which only the special sciences have "basic laws" or "axioms" while logic does not.[28] Beyond that, why did he consider his project to be "logical" even though it involved existential claims? Here a comparison to Frege is helpful again. Why did Frege considered his "Basic Law V" for classes (or value ranges) to be logical? He saw it as a "conceptual" and, therefore, "logical" truth (as opposed to an intuition- or perception-based geometric or scientific truth), it seems. Dedekind might well have assumed the same. Or again, we could take his talk of "abilities of the mind" very seriously and try to spell it out along Kantian lines, as indicated earlier.

So much for the "logical" constructions involved in Dedekind's approach. Earlier, I pointed out that a form of "abstraction" plays a core role for him as well. Here too one might, especially after Frege's criticisms, have expected the formulation of basic principles by him. But once again, Dedekind did not make explicit such principles; he only gave

25 Cf. Reck (forthcoming) for a further discussion of Frege's relation to Dedekind.

26 In Ferreirós (1999), a closely related "dichotomy principle" for sets is attributed to Dedekind.

27 Once he became aware of it, Dedekind was very interested in Schröder's work on logic, which remained in the Boolean tradition but also incorporated Dedekindian techniques.

28 Cf. Ferreirós (forthcoming) for this second suggestion.

some related hints (about "neglecting the special character of objects", "only retaining their distinguishability", etc.). To reconstruct Dedekind's logicism fully, we would thus have to add "abstraction principles" for him as well. Here again, Frege's approach, or a Fregean neo-logicist approach, might be compared profitably, especially with respect to the form of such principles.[29] Yet doing so still leaves us with the question of why we should consider those principles to be "logical".[30] Perhaps they too have to be seen as general "conceptual" truths (as opposed to intuition- or perception-based truths). But all of this is clearly in need of further clarification. In other words, many questions remain about the notion of "logic" involved.[31]

Conclusion: The Contested Notion of "Logic" and Its Mathematical Background

I assume contemporary readers will find it most natural to reconstruct Dedekind's foundational contributions along set-theoretic lines, or perhaps along category-theoretic lines, and in either case, ultimately in "formalist" terms. This is tempting especially if one rejects that "logic" can underwrite any existence assumptions, which have to be supplied by formalistically understood "axioms" instead. Yet Dedekind presented his project as one of showing that arithmetic is "part of logic". Moreover, several of his contemporaries, both on the mathematical side (Schröder, Hilbert) and on the philosophical side (Cassirer, Peirce), followed him in that characterization, at least initially. Similarly, Frege and Dedekind took themselves to be involved in parallel projects, even if Dedekind was not explicit enough about his basic laws, as Frege complained. For both Dedekind and Frege, the needed construction and abstraction principles were meant to be "logical" in the end; and for both, they involved general notions of function and set/class.

A broad, inclusive notion of "logic" is at play in both Dedekind's and Frege's forms of logicism, as I have argued. But this notion is not only hard to reconstruct in its details, it is also contested. And it is clearly in conflict with a notion according to which only statements "true in all domains" count as "logical", so that existence claims are ruled out from the start. Partly because of the antinomies of set theory, partly

29 Cf. Linnebo and Pettigrew (2014) for the kind of principles I have in mind. Note that they are modeled on neo-logicist abstraction principles (but with a "structuralist" twist). This may suggest ways of thinking about them as "logical", similar to Frege's attitude about his Basic Law V.

30 Cf. the discussion of "Hume's Principle" as "quasi-definitional" in neo-logicism.

31 A different argument might be that, since Dedekind abstraction is "structuralist", it involves a form of "permutation invariance" that qualifies it as "logical" (along Tarskian lines).

also because of the rise of formalism (as tied to Hilbertian axiomatics), that notion of logic became dominant in the 20th century. Yet from a historical point of view such a shift constitutes a narrowing of the sense of "logic". If we accept this narrowing, only the truths of first-order logic, or perhaps those of simple type theory, qualify in the end. All "logic" itself can provide, then, is a deductive framework for the mathematical sciences, while any existential assumptions have to be added as non-logical "axioms". One thing reconsidering Dedekind's logicism can show is that this view was not always taken for granted, i.e. it can remind us of an older, broader sense of "logic".

"Logic" in this broader sense grew out of certain mathematical developments in the 19th century, especially the switch from taking geometry as the basis for all of mathematics (Euclidean "geometricism"), to rethinking all "pure mathematics" first in "arithmetic" terms (the "arithmetization" of analysis and algebra) and then in purely "logical" terms. Both Frege and Dedekind were part of that movement. Frege's contributions have impacted philosophy more, especially in the analytic tradition, while Dedekind's are more relevant for mainstream mathematics, as his influence on axiomatic set theory and category theory illustrates.[32] In any case, it would be an impoverishment, both historically and philosophically, to forget the "logical" origins of such 20th-century theories, even if they are understood "formalistically" today. Finally, the question of how exactly either of them goes beyond "logic" seems worth reconsidering. For all we know, the described narrowing of our understanding of "logic", hidden as they are from the usual ahistorical perspectives, might become subject to change again.

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32 See also the recent rise of Homotopy Type Theory and the project of Univalent Foundations, which might be seen other successors of Dedekind's (and partly Frege's) "logicist" project.

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